Pictures of Monomial Ideals

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Why study ideals?

For now, think of ideals as sets of polynomials

- Solving Equations
  - Linear
  - Quadratic
  - Cubic
  - Higher degree?
- Systems of Equations
  - Several variables
  - Linear
  - Higher degree?

Ideals arise in Ring Theory

A **ring** $\boldsymbol{R}$ (commutative, with identity) is a set with the following properties:

- Closed under addition and multiplication
- Associative and commutative under addition and multiplication
- Additive identity (0)
- Additive inverses
- Multiplicative identity (1)
- May NOT have multiplicative inverses
  - If all nonzero elements do, it's called a **field**.

Examples of Rings

- $\mathbb{R}$, the set of real numbers
- $\mathbb{Q}$, the set of rational numbers
- $\mathbb{Z}$, the set of integers
- $\mathbb{Z}[x]$, polynomials in one variable with integer coefficients
- $\mathbb{R}[x, y]$, polynomials in two variables with real coefficients

Ideals

An **ideal** $\boldsymbol{I}$ is a subset of a ring R

satisfying the following property:

- **If $\boldsymbol{f}, \boldsymbol{g}$ are in $\boldsymbol{I}$, then $\boldsymbol{af} + \boldsymbol{bg}$ is in $\boldsymbol{I}$ for any $\boldsymbol{a}, \boldsymbol{b}$ in $\boldsymbol{R}$.**
- That is, I is closed under linear combinations with coefficients in the ring.
  - Closed under addition
  - Closed under “scalar” multiplication

Examples of Ideals

- $R = \mathbb{Z}, I = (5)$

  $$= \{5a : a \in \mathbb{Z}\}$$

- $R = \mathbb{R}[x, y], I = (x^2 - xy,\ 3x + y)$

  $$= \{a(x^2 - xy) + b(3x + y) : a, b \in R\}$$

- $R = \mathbb{R}[x, y], I = (x^2,\ xy^3,\ y^5)$

  $$= \{ax^2 + bxy^3 + cy^5 : a, b, c \in R\}$$

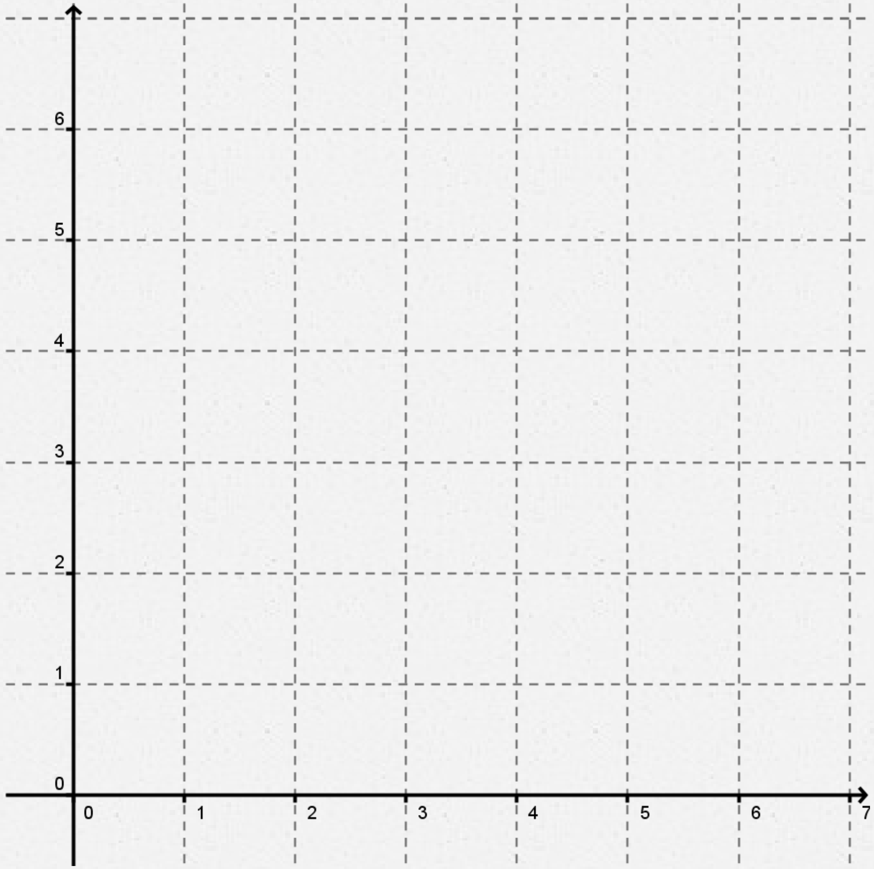
  - Each generator is a monomial, a single term

Rings mimic the Integers

- Prime factorization / Primary decomposition
  - In $\mathbb{Z}$, factor 200
  - In $\mathbb{R}[x, y]$, factor $x^4y - x^3y^2$
  - What about (200) and (x^2, xy^3, y^5)?

- Modular arithmetic / Quotient rings
  - $\mathbb{Z}/(5) = \{a + (5) : a \in \mathbb{Z}\}$
  - $\mathbb{R}[x, y]/(x^2, xy^3, y^5) =$?
    - Allows us to find the dimension or "size"

Pictures!



$$I = (x^2, xy^3, y^5)$$

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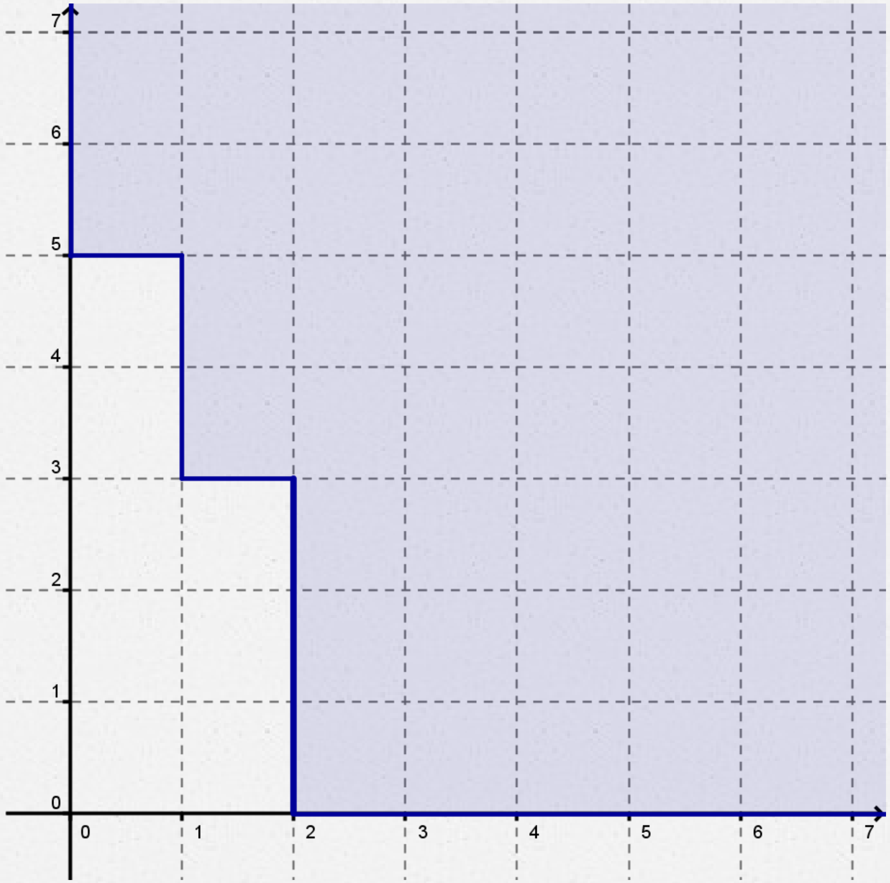
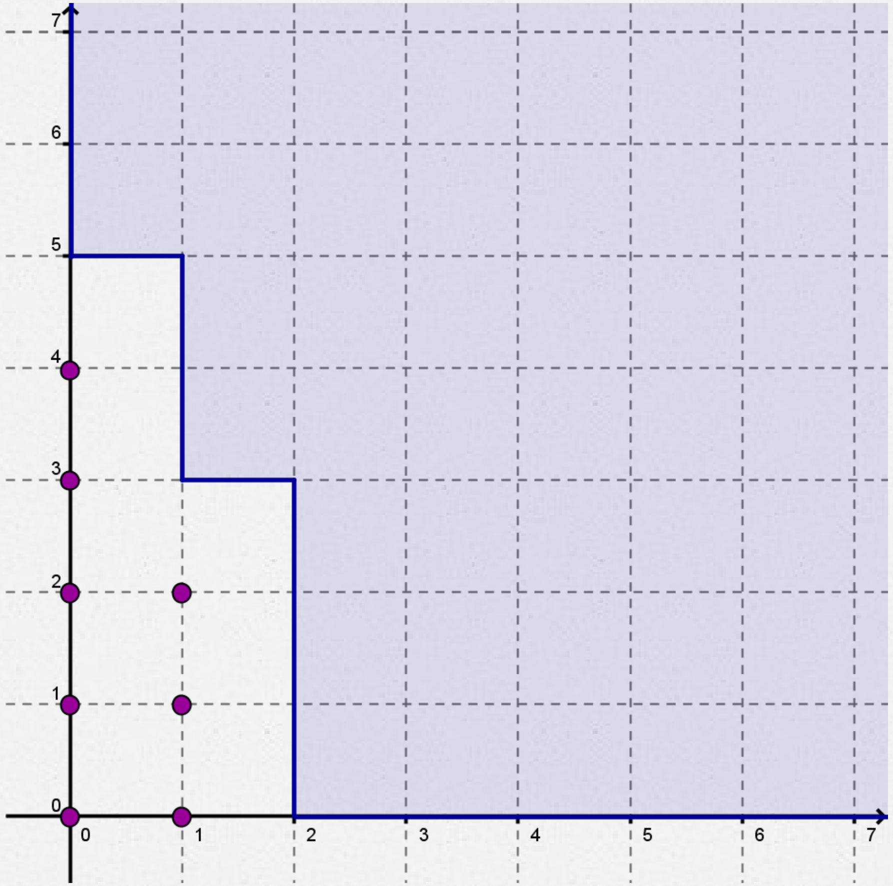
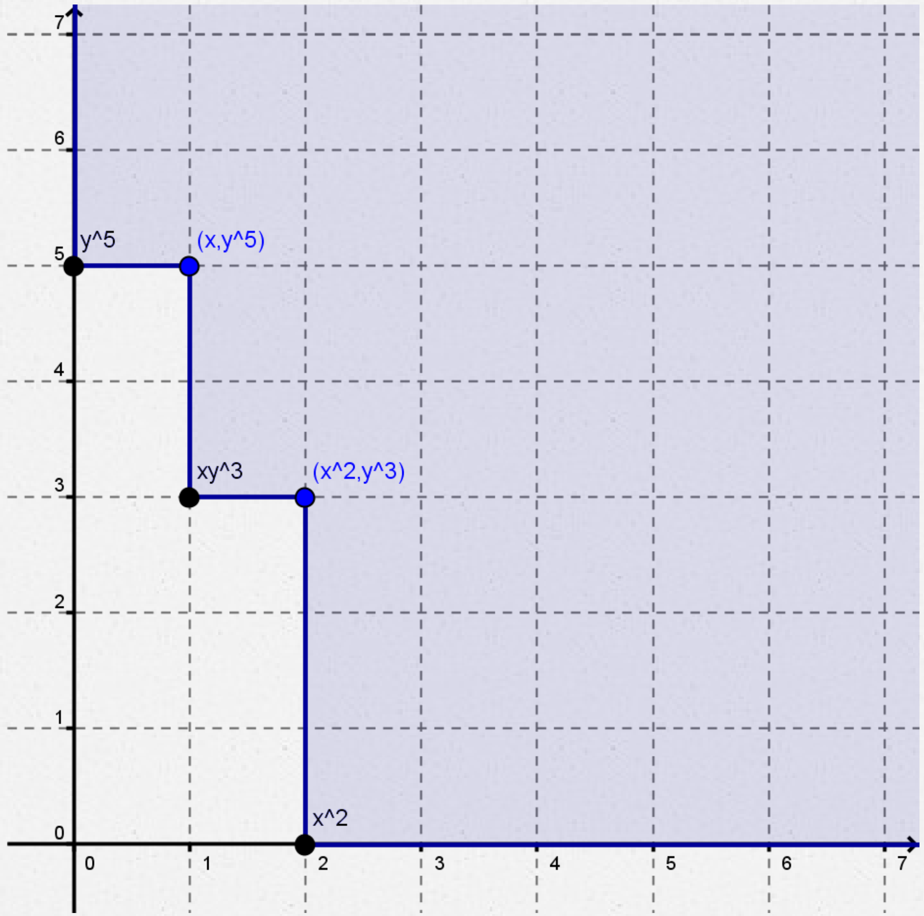


Dimension of R/I?



Primary Decomposition



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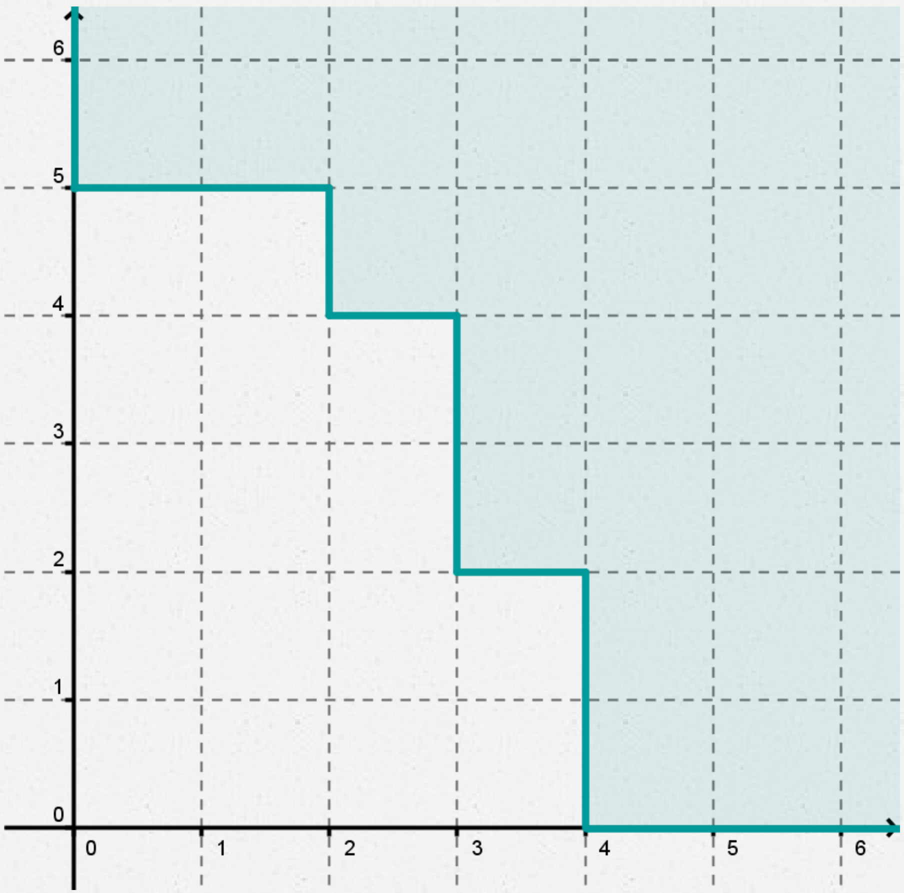
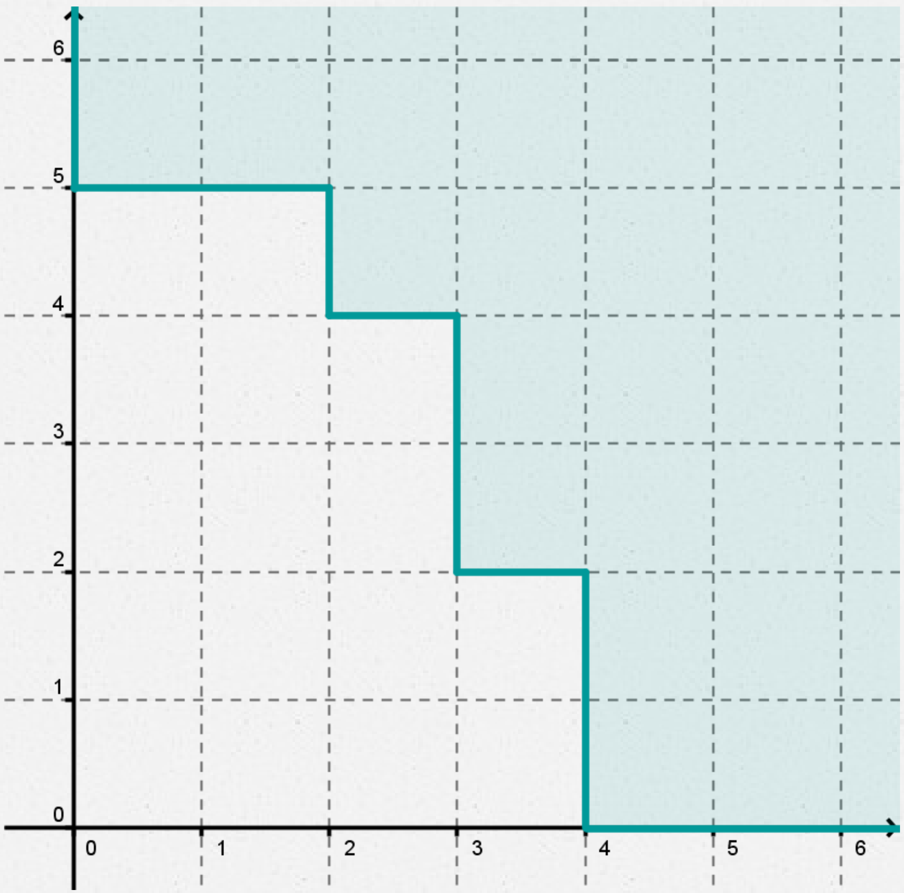


Primary Decomposition?



What if $R = \mathbb{R}[x, y, z]$?

Let $I = (x^3, y^4, z^2, xy^2z)$.

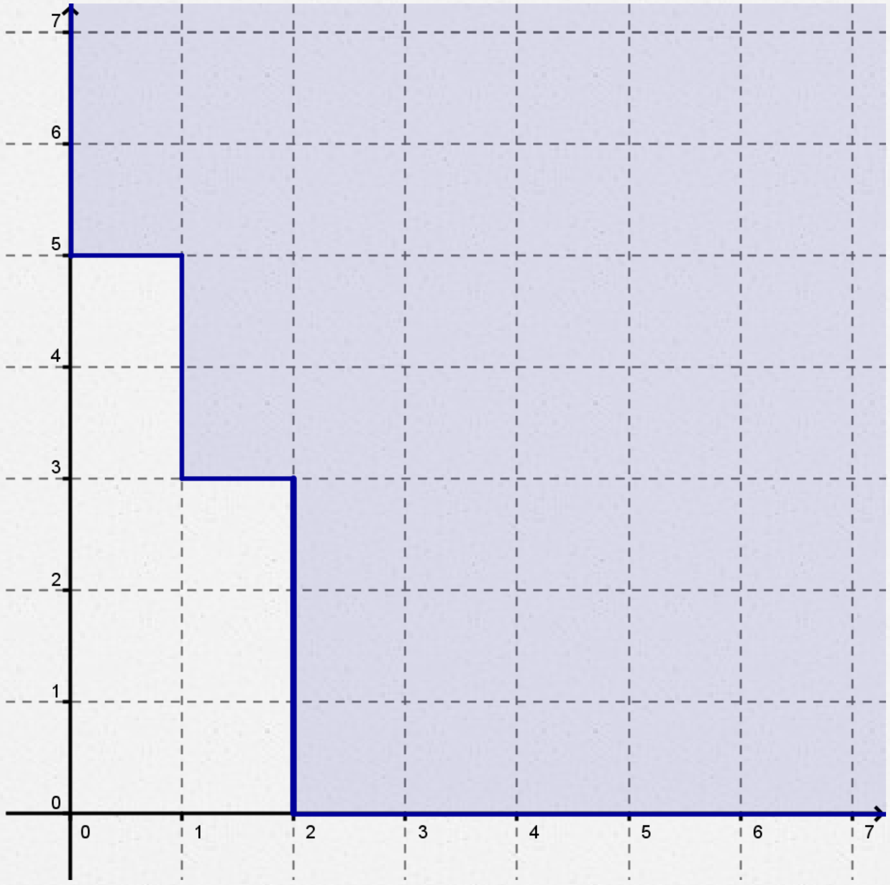
- Diagram: think R/I
- Dimension of R/I?
- Primary Decomposition?

Colon Ideals

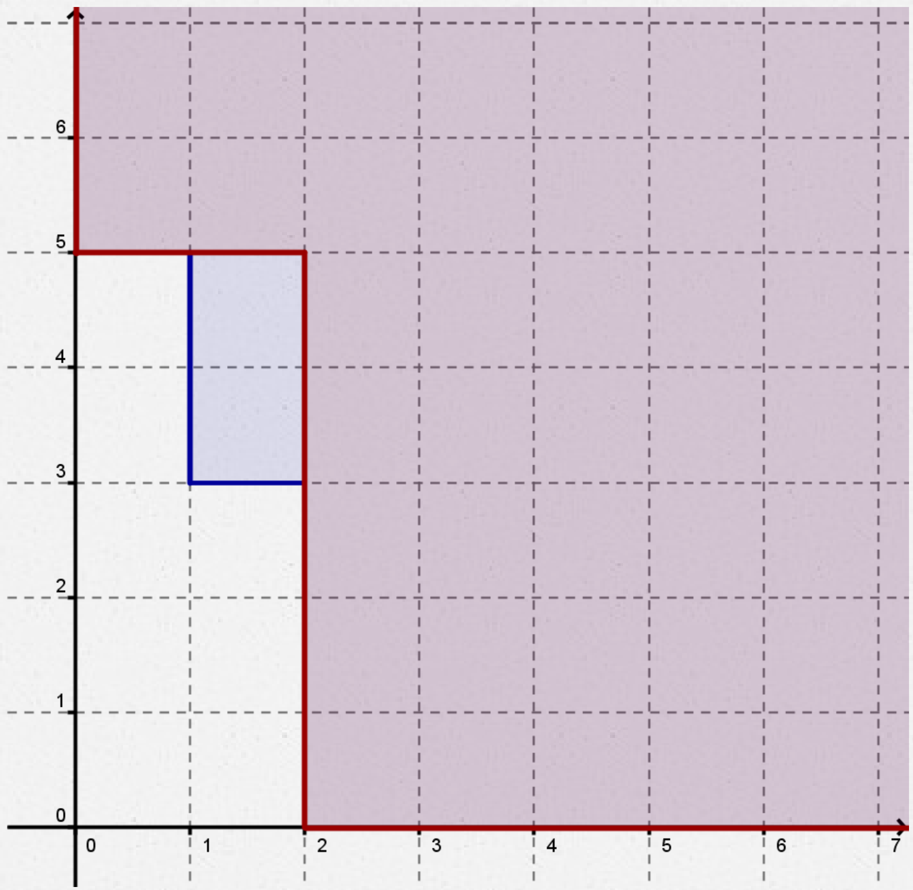
Let H and I be ideals in a ring R with H contained in I.

- Then $H: I = \{a \in R : aI \subseteq H\}$
- That is, $H: I$ is the set of elements of the ring which move all elements of I into H.

$I = (x^2, xy^3, y^5)$ $H = (x^2, y^5)$



$I = (x^2, xy^3, y^5) \qquad H = (x^2, y^5)$



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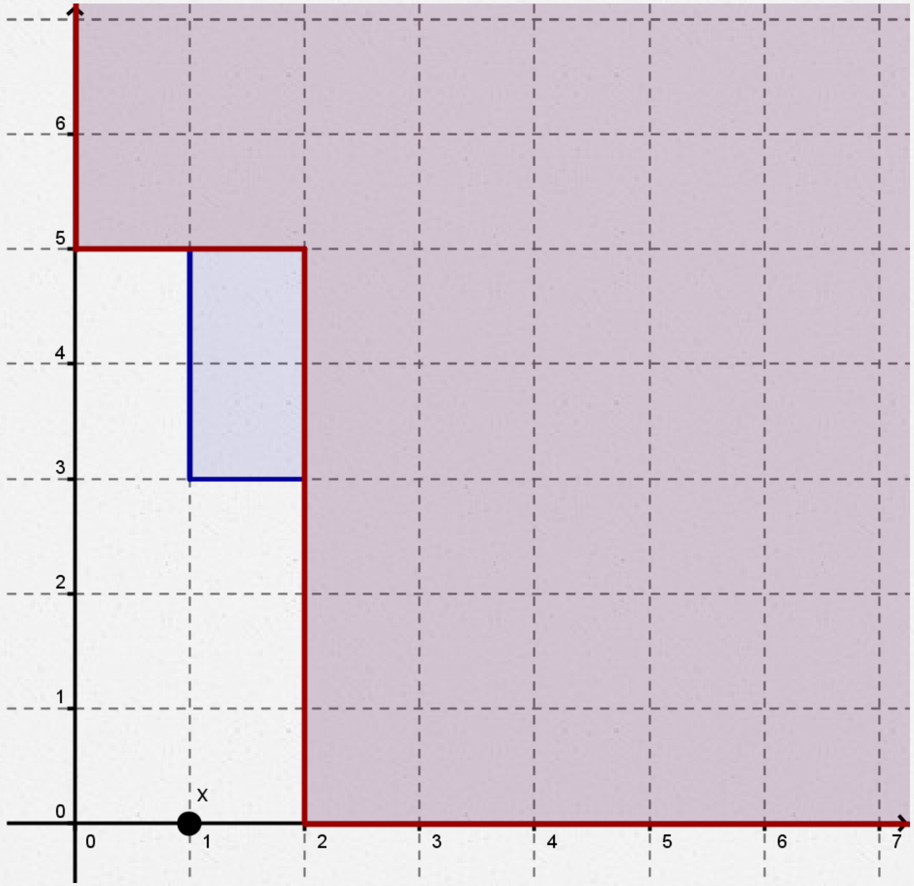
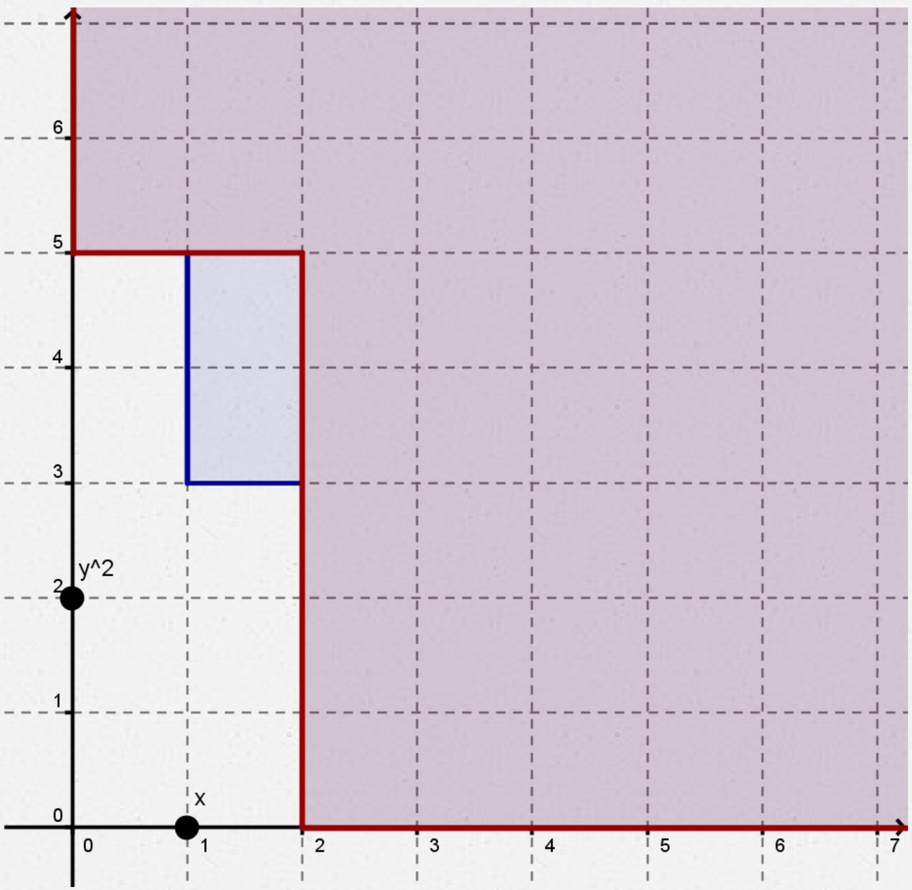
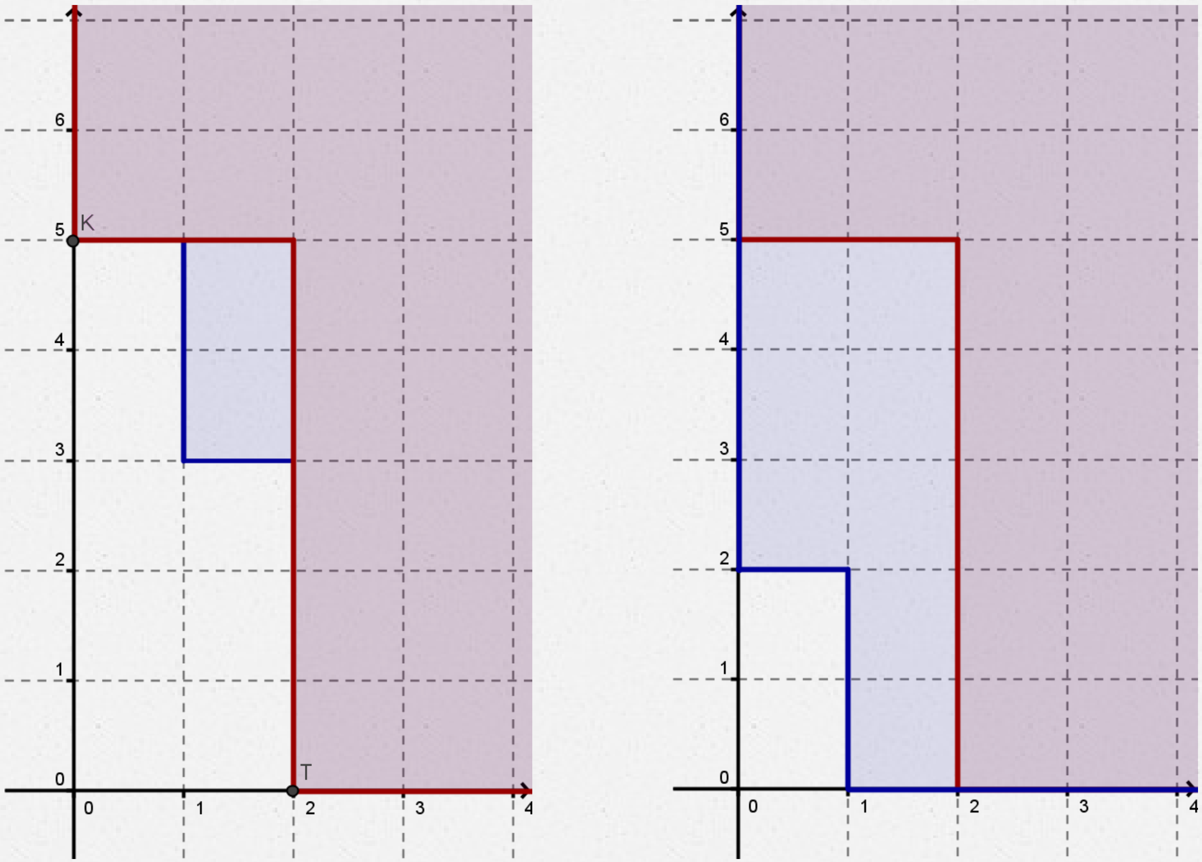


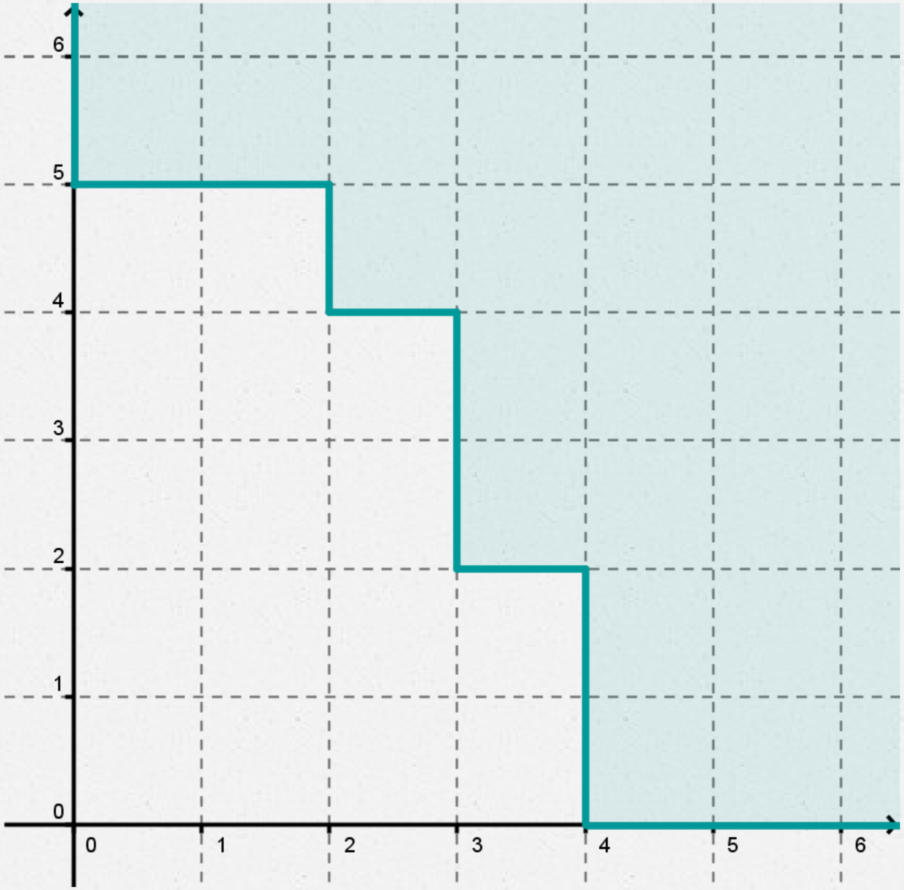
$$I = (x^2, xy^3, y^5) \qquad H = (x^2, y^5)$$



$$H: J = (x, y^2)$$



$$I = (x^4, x^3y^2, x^2y^4, y^5) \quad H = (x^4, y^5)$$



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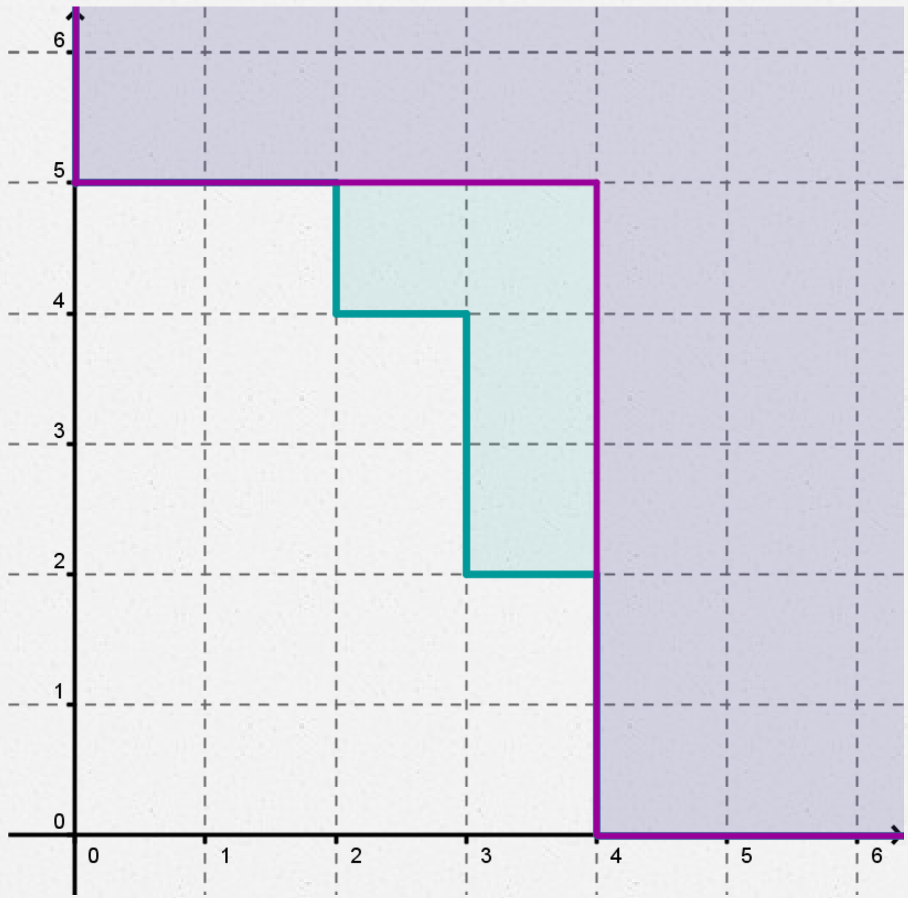

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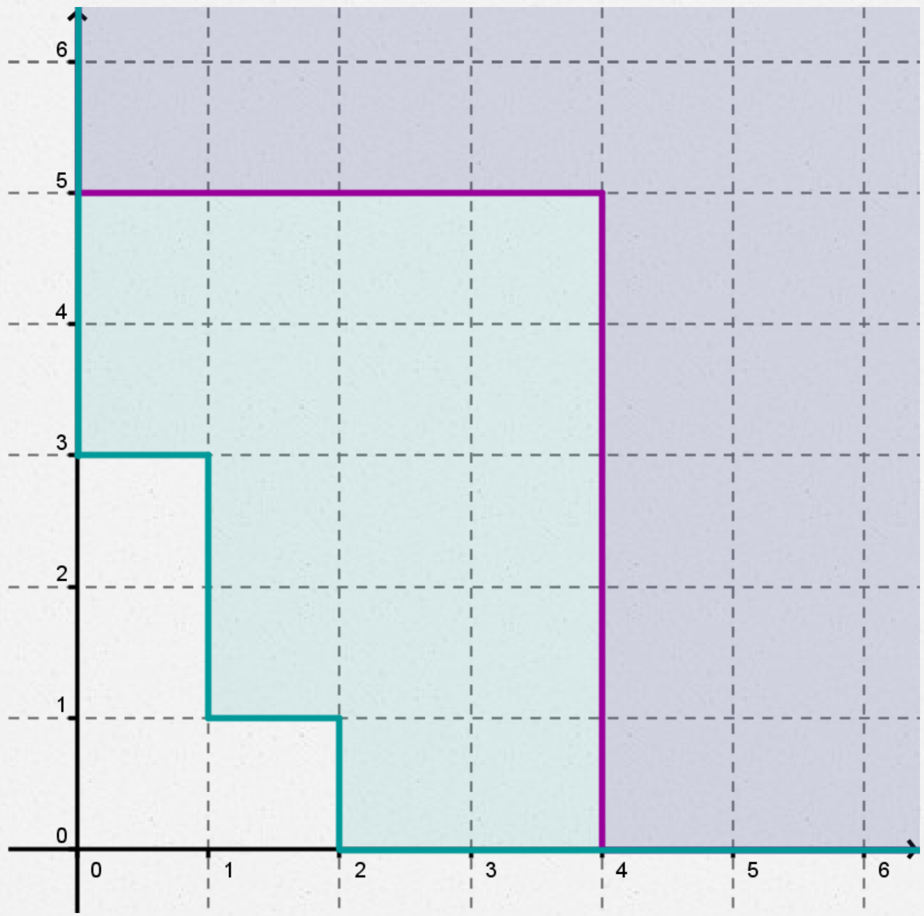


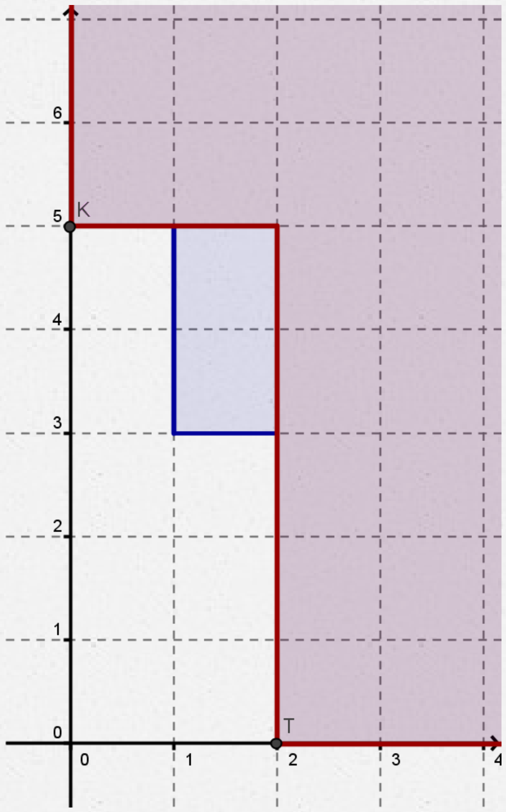
Reductions of Ideals

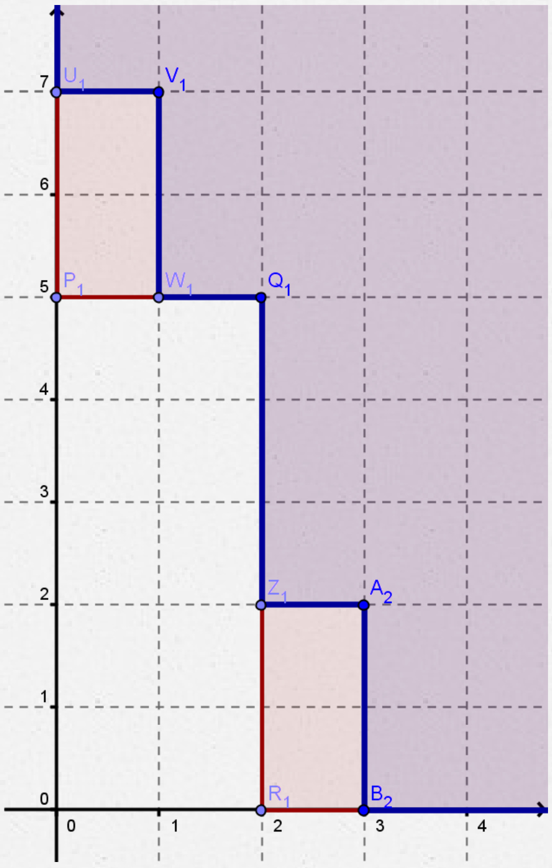
- **Reductions** are simpler ideals contained in a larger ideal with similar properties.
  - "Simpler" usually means fewer generators.
  - Most minimal reductions are not monomial, but their intersection is.

The Core of an Ideal

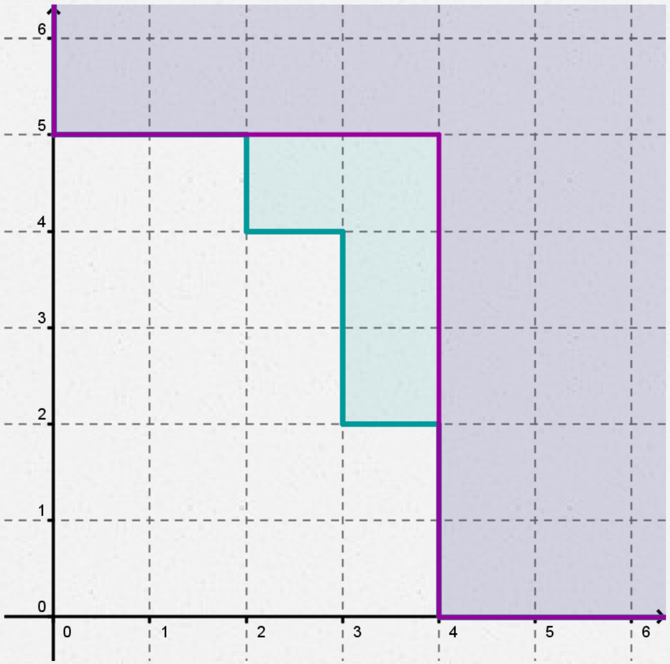
- The **core** of an ideal I is the intersection of all reductions of I.
  - If I is monomial, so is $\mathrm{core}(I)$.
  - Cores have symmetry similar to colon ideals.

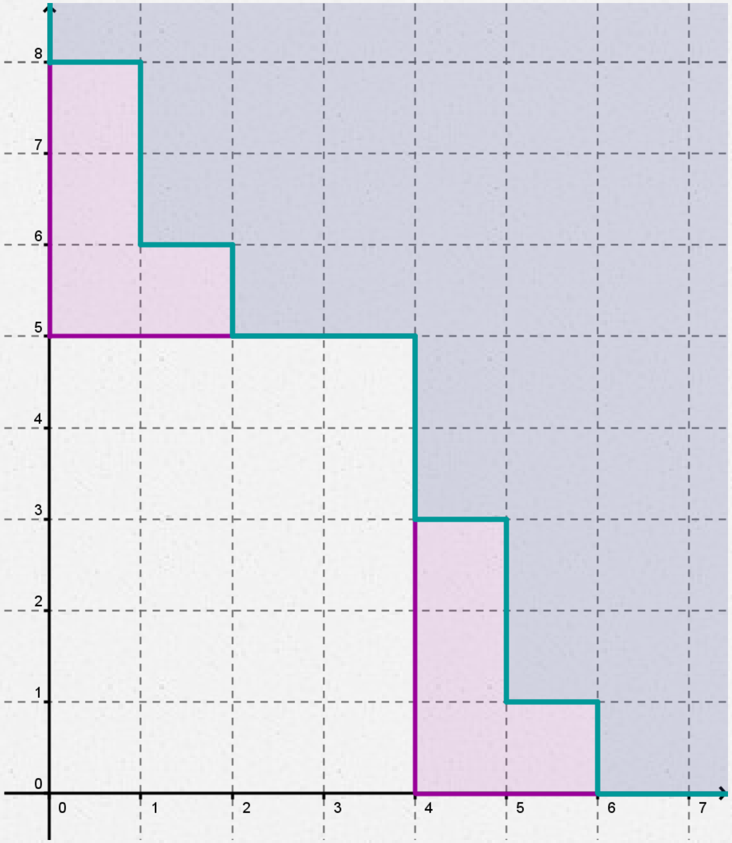
$I = (x^2, xy^3, y^5)$ $\mathrm{core}(I)$



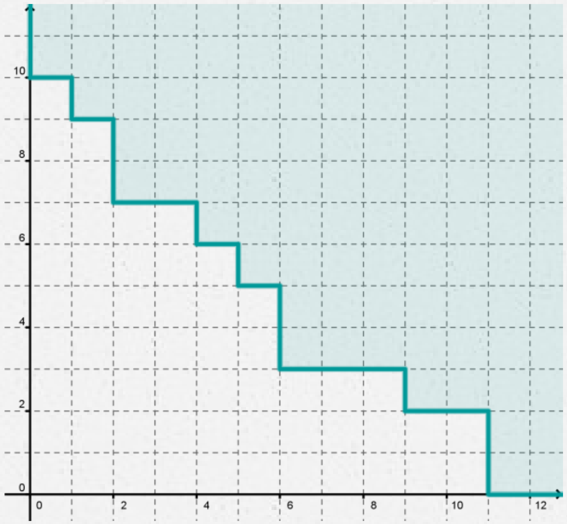


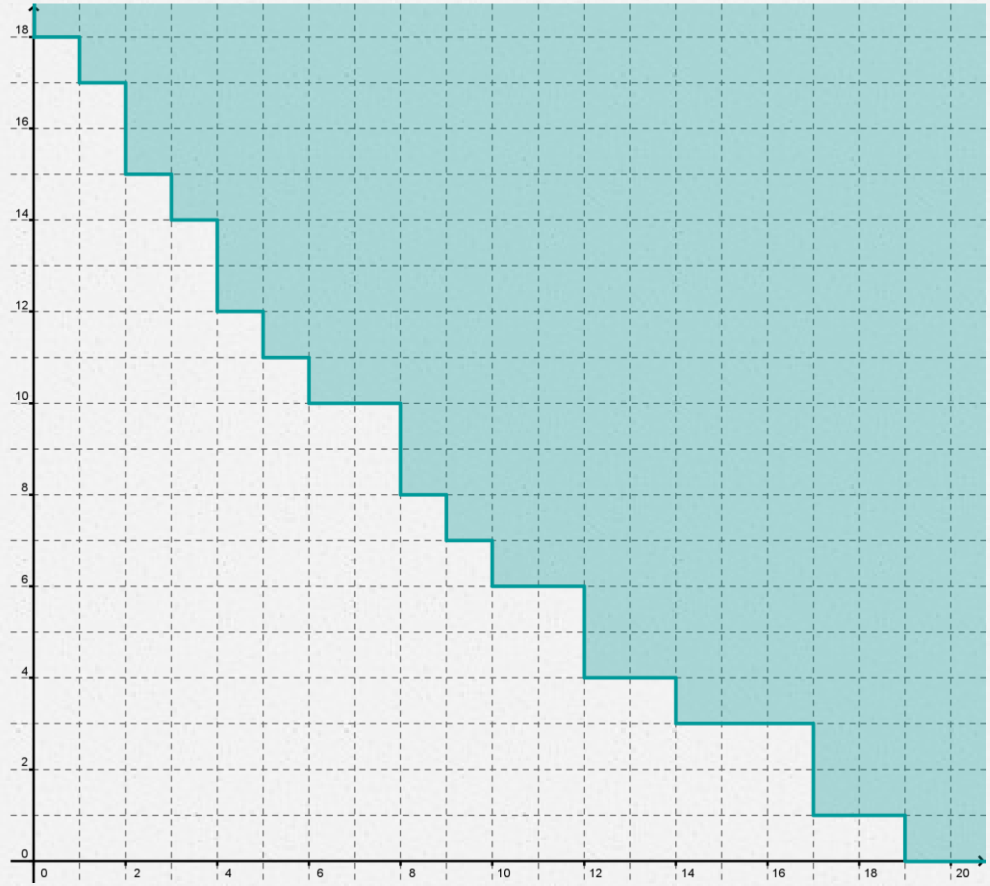
$I = (x^4, x^3y^2, x^2y^4, y^5) \quad \text{core}(I)$





$$I = (x^{11}, x^9y^2, x^6y^3, x^5y^5, x^4y^6, x^2y^7, xy^9, y^{10})$$





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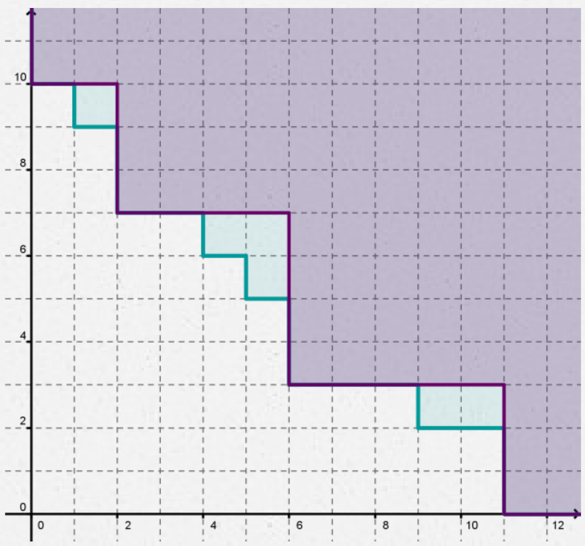
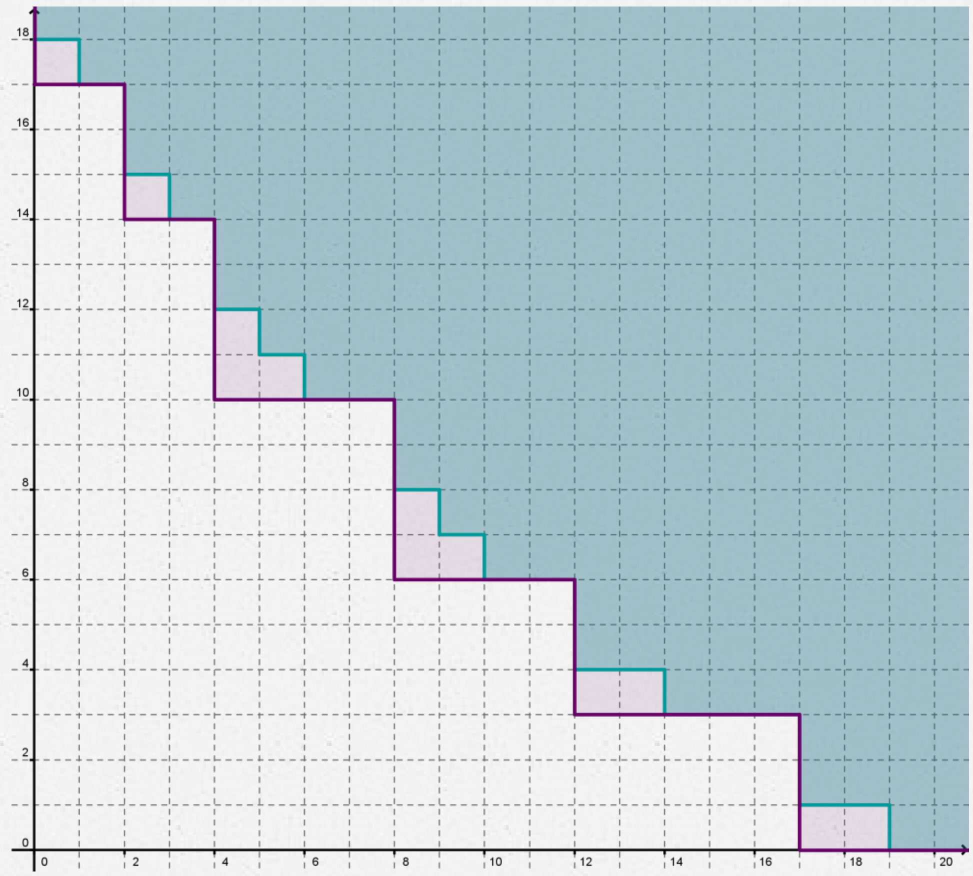
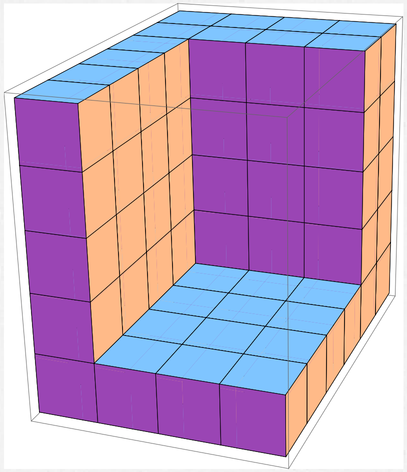
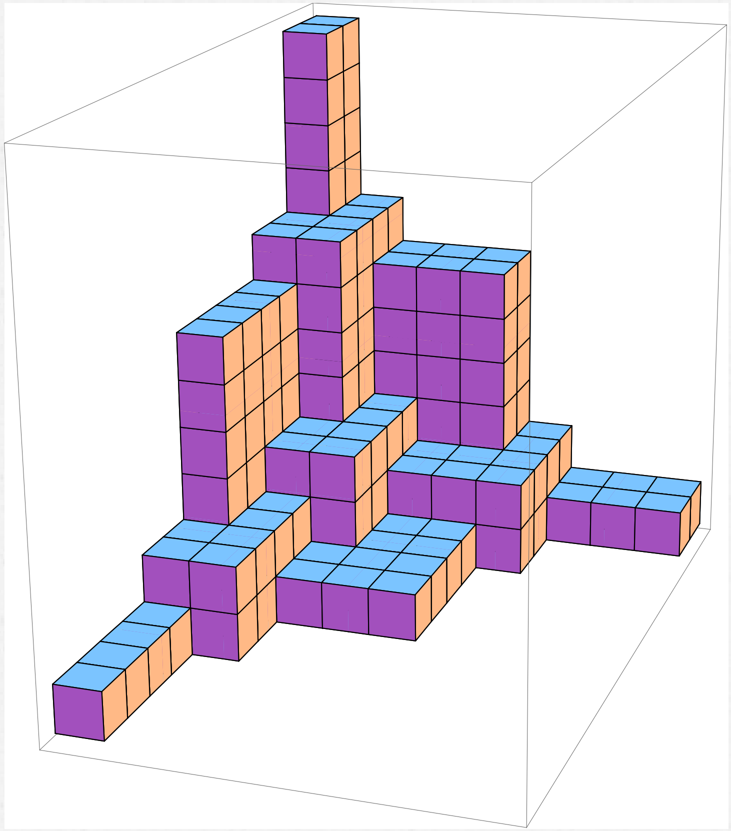




$I = (x^6, y^4, z^5, x^2yz)$ $\quad$ $\mathrm{core}(I)$





Why study monomial ideals?

- Can reduce more complicated ideals to monomial ideals with similar properties (Gröbner basis theory).
- Monomial ideals can be studied with combinatorial methods, not just algebraic.
- They are algorithmic, easy to program.