The core versus the adjoint of a monomial ideal

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AMS Sectional Meeting
Lexington, March 27, 2010

What is the core?

Let $J \subseteq I$ be ideals in a Noetherian ring R.

Definition (Northcott and Rees)

- *J is a **reduction** of I if $JI^r = I^{r+1}$ for some r.*
- *$r_J(I)$: smallest such r, called the **reduction number of I with respect to J**.*
- *$r(I)$: minimum $r_J(I)$ varying over all minimal reductions J, called the **reduction number of I**.*

Equivalently,

- J is a reduction of I if $R[Jt] \subseteq R[It]$ is integral, where $R[It] = \bigoplus_{i \geq 0} I^i t^i$.
- J is a reduction of I if $\overline{J} = \overline{I}$.

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What is the core?

Let J be a minimal reduction of I (with respect to inclusion):

$$J \subset I \subset \bar{I}$$

Generally, minimal reductions are highly non-unique.

Definition (Rees and Sally)

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Definition (Lipman)

Let R be a regular domain. The **adjoint** *of an ideal I in R is the ideal*

$$\operatorname{adj}(I) = \bigcap_{V \in D(R)} IJ_{V/R}^{-1},$$

where $D(R)$ is the set of divisorial valuations with respect to R and $J_{V/R}$ is the Jacobian ideal of V over R.

Fact: $\operatorname{adj}(I)$ is an integrally closed ideal of R.

Remark: When both notions are defined, $\operatorname{adj}(I)$ is equal to the *multiplier ideal* associated to I.

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How are the core and adjoint related?

Let I be an $\mathfrak{m}$-primary ideal in a RLR $(R, \mathfrak{m})$ of dimension d.

Briançon-Skoda Theorem: $\overline{I^d} \subset \text{core}(I)$.

Theorem: (Lipman) $\overline{I^d} \subset \text{adj}(I^d) \subset \text{core}(I)$.

Question 1: When is $\text{adj}(I^d) = \text{core}(I)$?

Question 1a: Is an integrally closed core sufficient for $\text{core}(I) = \text{adj}(I^d)$?

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Connection to Kawamata's Conjecture

Kawamata's Conjecture: Assume $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ is smooth and $\mathcal{L}$ is an ample line bundle on X with $\mathcal{L} \otimes \omega_X^{-1}$ ample. Then $H^0(X, \mathcal{L}) \neq 0$.

Theorem: (Hyry, Smith) Let $R = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n)$ and $d = \dim R$. Fix $N \gg 0$ and let $I = R_{\geq N}$. Then Kawamata's conjecture holds if and only if $\mathrm{gradedcore}(I\omega_R) = \mathrm{adj}(I^d \omega_R)$.

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History of the question

Theorem: (Huneke, Swanson) Let I be an integrally closed ideal in a RLR of dimension $d = 2$. Then $\operatorname{core}(I) = \operatorname{adj}(I^2)$.

Special case:

- There exist integrally closed monomial ideals I in $d = 3$ with $\operatorname{core}(I) \neq \operatorname{adj}(I^3)$.
- When $d = 2$, $R[It]$ satisfies Serre's condition R_1 and is CM and has only rational singularities.
- (Polini, Ulrich) $R[It]$ has only rational singularities $\implies \operatorname{core}(I) = \operatorname{adj}(I^d)$.

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Monomial ideals

Let I be a monomial ideal in $R = k[x_1, \ldots, x_d]$.

Fact: $\operatorname{core}(I)$ and $\operatorname{adj}(I)$ are monomial ideals.

Theorem (Howald, generalized by Hübl and Swanson)

$$\operatorname{adj}(I) = (\mathbf{x}^{\mathbf{e}} \mid \mathbf{e} \in \mathbb{N}^d,\ \mathbf{e} + (1, \ldots, 1) \in \mathbf{NP}^{\circ}(I))$$

Question 2: Is there a combinatorial description for the core of a monomial ideal?

Question 2a: What is the shape of the core of a monomial ideal?

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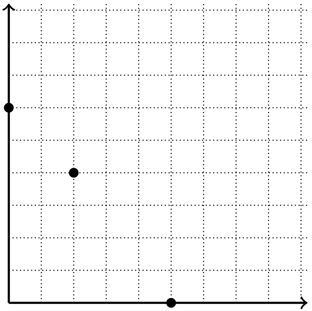
Example: $I = (x^5, x^2y^4, y^6)$ in $k[x, y]$



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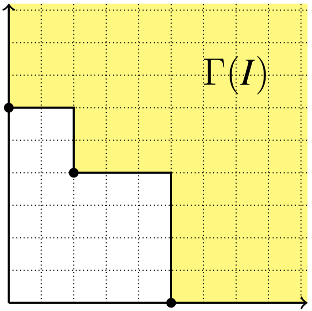

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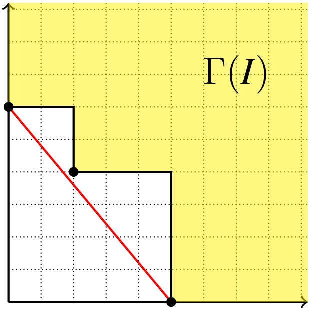

$\mathbf{NP}(I) =$ the convex hull in $\mathbb{R}^2$ of $\Gamma(I)$.

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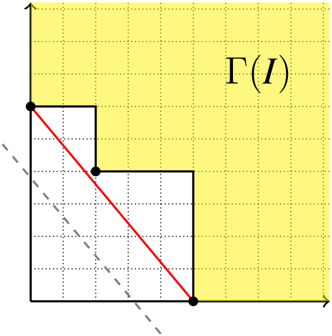

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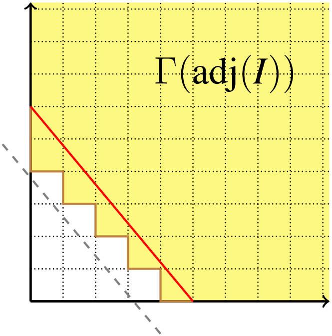

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Conjecture, based on a question of Huneke

Assumptions Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k of characteristic zero with unique homogeneous maximal ideal $\mathfrak{m}$. Let I be an $\mathfrak{m}$-primary monomial ideal of R.

Conjecture

Under the Assumptions, $\operatorname{core}(I) = \operatorname{adj}(I^d)$ *if and only if* $\operatorname{core}(I)$ *is integrally closed.*

Recent result:

Theorem: (Kustin, Polini, Ulrich) Let $I \subseteq k[x, y]$ be generated by forms of the same degree. Then $\operatorname{core}(I) = \operatorname{adj}(I^2)$ if and only if $\operatorname{core}(I)$ is integrally closed.

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The conjecture holds if

Case 1: I has a d-generated monomial reduction.

Case 2: $d = 2$ and either $\operatorname{core}(I) = \operatorname{core}(\bar{I})$ or I is not "too close" to $\bar{I}$.

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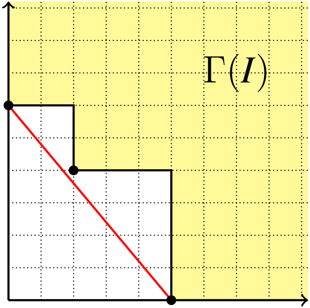
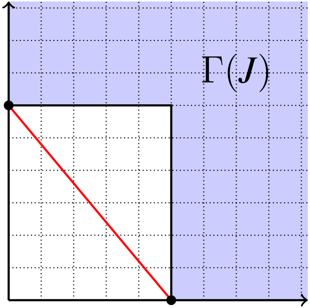
Translational symmetry of the core in Case 1

Example: $I = (x^5, x^2y^4, y^6)$. $J = (x^5, y^6)$ is a reduction of I.





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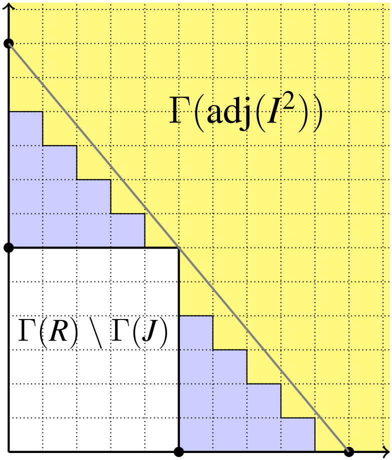

$$J \supset \mathrm{core}(I) \supset \mathrm{adj}(I^2)$$

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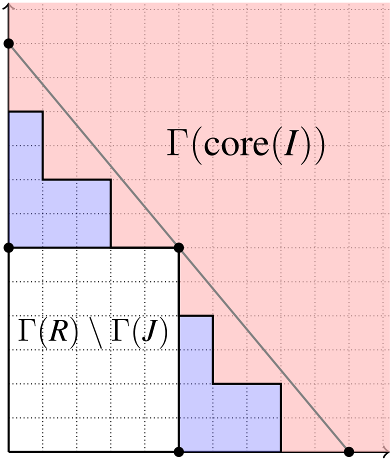

$$J \supset \mathrm{core}(I) \supset \mathrm{adj}(I^2)$$

$$\mathrm{core}(I) = (x^8, x^6y^2, x^5y^4, x^3y^6, xy^8, y^{10})$$

A generalization of $J = (x^a, y^b)$

Assume $d = 2$.

Let $\mathbf{a}_0, \ldots, \mathbf{a}_n$ minimally determine $\mathbf{NP}(I)$, where $\mathbf{a}_i = (a_i, b_i)$, $a_0 > \ldots > a_n = 0$ and $0 = b_0 < \ldots < b_n$.

We want a monomial ideal C with $C \supset \mathrm{core}(I) \supset \mathrm{adj}(I^2)$.

Proposition

Let I and $\mathbf{a}_0, \ldots, \mathbf{a}_n$ be as above.
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A generalization of $J = (x^a, y^b)$

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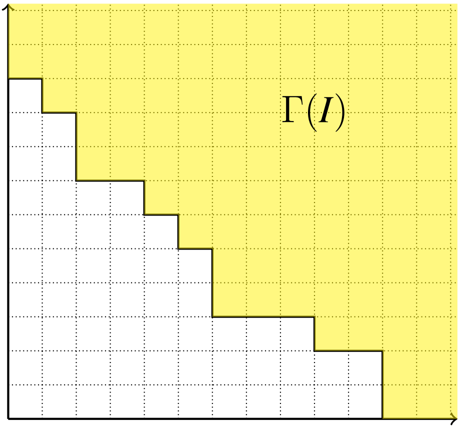

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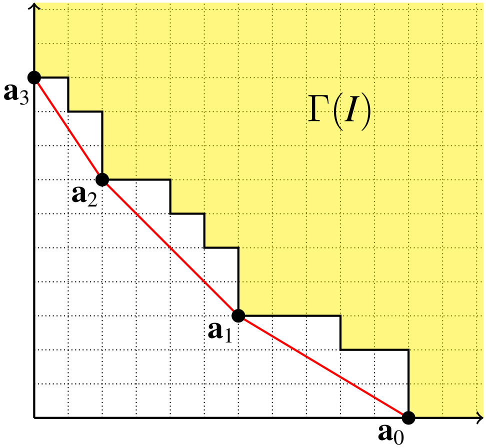

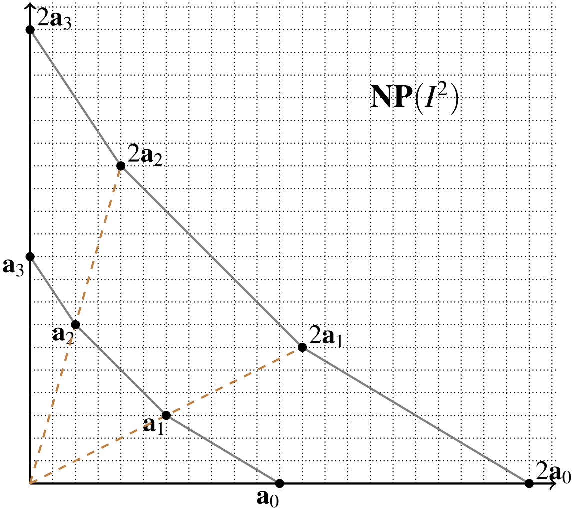

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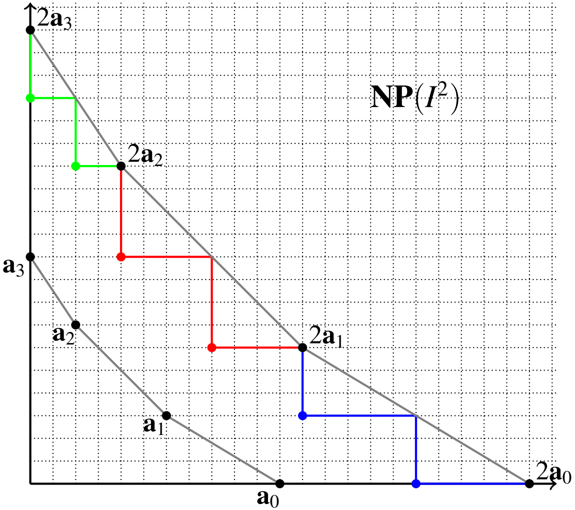

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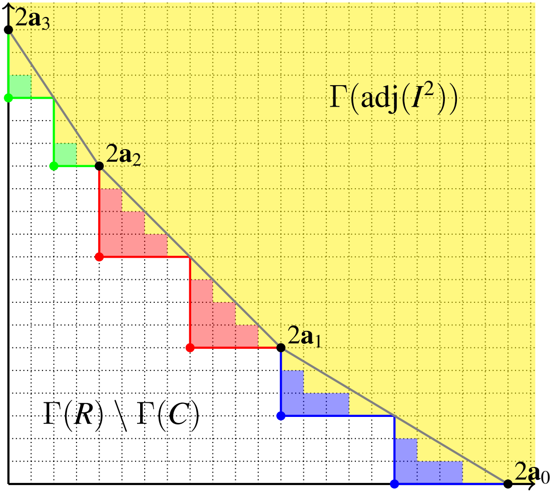

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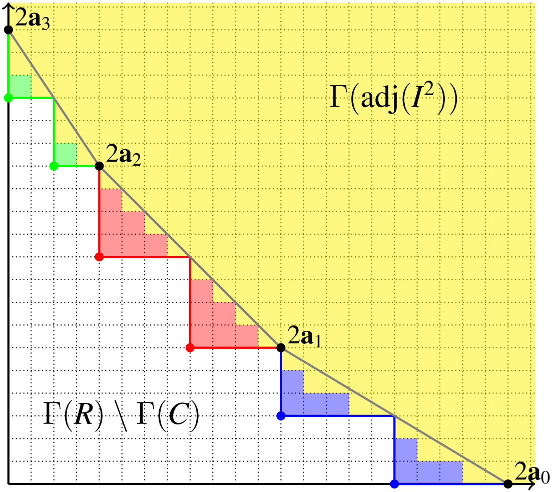
Does $\mathrm{core}(I)$ have local translational symmetry?



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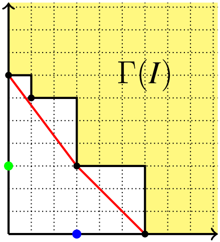

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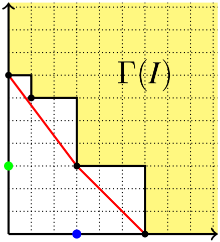

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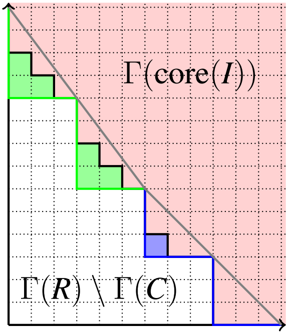

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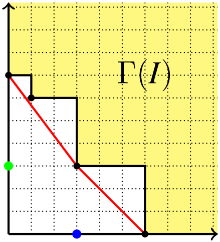

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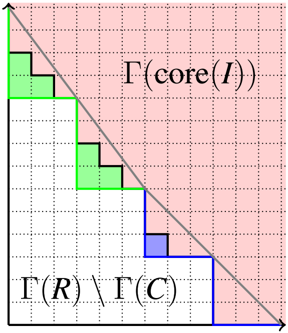

When IS there local translational symmetry?

The canonical module and the core

Assumptions 2: Let I be an m-primary monomial ideal of $k[x,y]$, and let J be a locally minimal reduction of I with $r = r_J(I)$.

Theorem: (Polini, Ulrich) Given Assumptions 2, $\mathrm{core}(I) = J^{r+1} : I^r = [\omega_{R[It,t^{-1}]}]_2$ and $J^r : I^r = [\omega_{R[It,t^{-1}]}]_1$.

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Translational symmetry in dimension 2

Proposition

Let I and J satisfy Assumptions 2. Then there is a unique ideal L between I and $\bar{I}$ which is

(a) *the largest ideal with $[\omega_{R[Lt,t^{-1}]}]_1 = [\omega_{R[It,t^{-1}]}]_1$;*
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Further, L is monomial, $L = J : (J^r : I^r)$, and $\operatorname{core}(L) = J(J : L) = J(J^r : I^r)$.

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Why?

Locally, J is a complete intersection and $J : L$ is a link.

We can use linkage and the Hilbert-Burch matrix to show $\Gamma(J : L)$ has reflectional symmetry with respect to $\Gamma(L)$.

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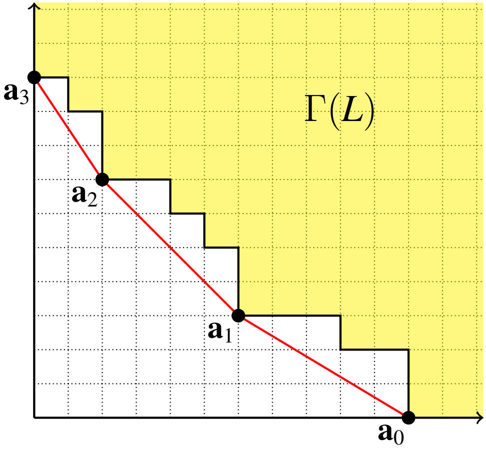
Reduction number one ideals and $J : L$



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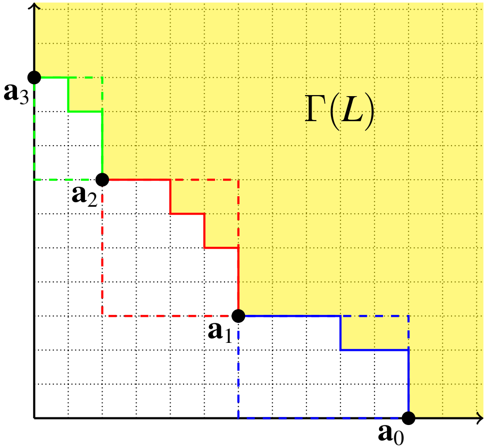

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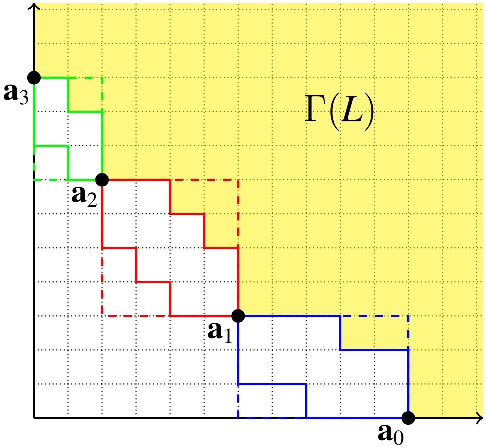
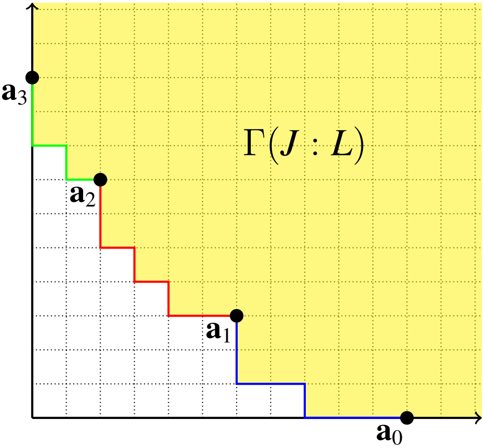
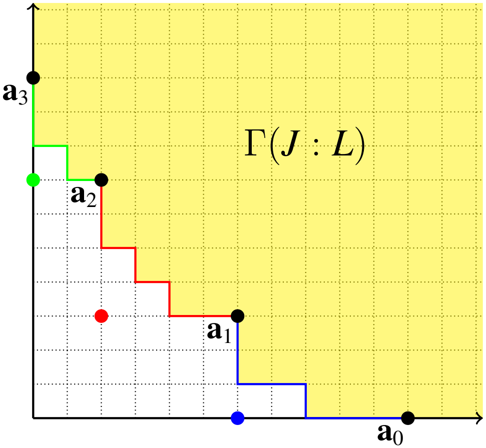
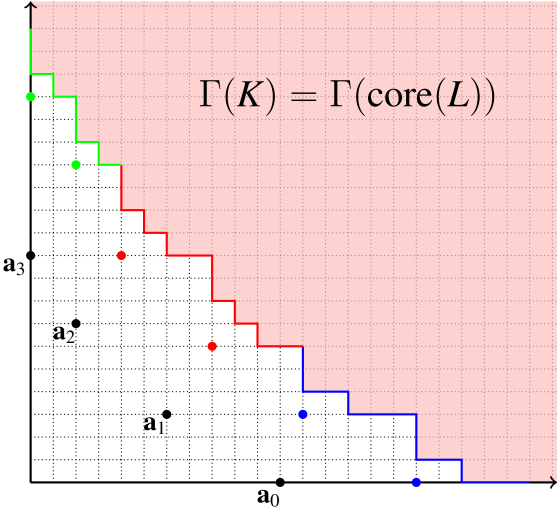


Proposition: This process gives us exactly $\Gamma(J:L)$.



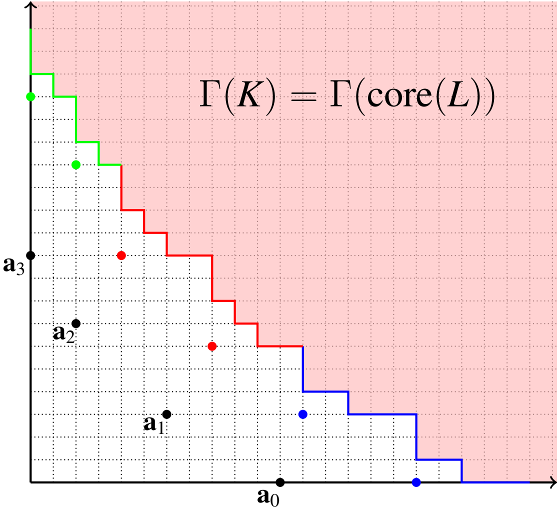
Define $K = \sum_{i=1}^{n} (\mathbf{x}^{\mathbf{a}_{i-1}}, \mathbf{x}^{\mathbf{a}_i})[(x^{a_i} y^{b_{i-1}}) \cap (J : L)]$.





Proposition:
$K = \mathrm{core}(L)$.

Idea: Multiplying $J : L$ by J results in duplicating each section of $\Gamma(J : L)$ to get $\Gamma(J(J : L)) = \Gamma(\mathrm{core}(L))$.



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Proof idea: Use the containment $C \supseteq \operatorname{core}(I) \supseteq \operatorname{core}(L)$ and the local translational symmetry of $\Gamma(\operatorname{core}(L))$.

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