The core and the adjoint: A condition for equality

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Assumptions and questions of research

Assumptions: Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k of characteristic zero with $\mathfrak{m} = (x_1, \ldots, x_d)$. Let I be an $\mathfrak{m}$-primary ideal of R ($\mathfrak{m}^n \subseteq I$ for some n).

Question 1: When is core of an ideal equal to the adjoint of its *d*-th power?

Question 2: Is there a combinatorial description for the core of a monomial ideal?

Background

The core

The intersection of all reductions

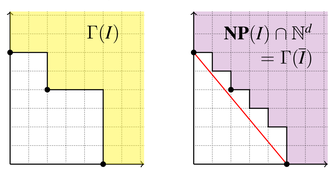
The **integral closure** of I is the ideal $\bar{I} = \{z \in R \mid z^n + a_1 z^{n-1} + \ldots a_n = 0, a_i \in I^i\}$.

Monomial case: Let I be monomial.

- I is characterized by its exponent set $\Gamma(I) = \{\underline{a} \in \mathbb{N}^d \mid \underline{x}^{\underline{a}} \in I\}$.
- The Newton Polyhedron $\mathbf{NP}(I)$ is given by the convex hull in $\mathbb{R}^d$ of $\Gamma(I)$.

Fact: I monomial $\Rightarrow \bar{I}$ is monomial and $\Gamma(\bar{I}) = \mathbf{NP}(I) \cap \mathbb{N}^d$.

Example: In $k[x,y]$, let $I = (x^5, x^2y^4, y^6)$. Then $\bar{I} = (x^5, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6)$.



A **reduction** of I is an ideal $J \subseteq I$ such that, equivalently,

- $\bar{J} = \bar{I}$;
- There exists an r with $JI^r = I^{r+1}$;
- $R[Jt] \hookrightarrow R[It]$ is module-finite, where $R[It] = \bigoplus_{i \geq 0} I^i t^i$ is the Rees algebra of I.

Let $\mathcal{M}(I)$ be the set of minimal reductions with respect to inclusion.

Fact: Then $\mathcal{M}(I)$ is the set of d-generated reductions of I.

Problems: *(a)* $|\mathcal{M}(I)| = \infty$.
(b) I monomial $\not\Rightarrow$ a monomial $J \in \mathcal{M}(I)$ exists.

So, Rees and Sally defined the core:

Definition: $\operatorname{core}(I) = \bigcap_{J \in \mathcal{M}(I)} J$.

Fact: I monomial $\Rightarrow \operatorname{core}(I)$ is monomial.

The adjoint

A tool for studying singularities

The adjoint of I, also called the **multiplier ideal** associated to I, measures "how bad" the singularities of $V(I)$ are.

Three contexts for the adjoint:

- Algebra: valuations and modules of differentials
- Analysis: integrals of plurisubharmonic functions
- Geometry: resolutions of singularities

Definition: $\operatorname{adj}(I) = \bigcap_{V \in D(R)} IV : J_{V/R}$.

- $D(R)$ is the set of discrete valuation rings between R and $\operatorname{Quot}(R)$.
- $J_{V/R}$ is the **Jacobian ideal** of V over R, where $J_{V/R} = \operatorname{Fitt}_0 \Omega_{V/R}$.

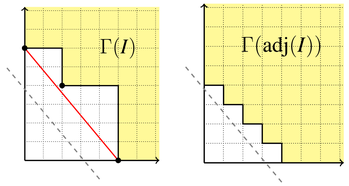
Fact: $\operatorname{adj}(I)$ is an integrally closed ideal of R, and $\operatorname{adj}(I) = \operatorname{adj}(\bar{I})$.

Monomial case: Howald's combinatorial description:

Theorem: I monomial $\Rightarrow \operatorname{adj}(I) = (\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d, \underline{e} + (1, \ldots, 1) \in \mathbf{NP}^\circ(I))$.

- $\mathbf{NP}^\circ(I)$ is the interior of $\mathbf{NP}(I)$.

Example: Let $I = (x^5, x^2y^4, y^6)$. Then $\operatorname{adj}(I) = (x^4, x^3y, x^2y^2, xy^3, y^4)$.



- $\mathbf{NP}(I^d) = (\lambda_1 + \ldots + \lambda_d \mid \lambda_i \in \mathbf{NP}(I))$, making $\operatorname{adj}(I^d)$ easy to compute.

History of core and adjoint relationship

The theorem below was the original connection made between the core and the adjoint.

Theorem: (Lipman, generalized by Ulrich) Let I be an ideal in $k[[x_1, \ldots, x_d]]$. Then $\overline{I^d} \subseteq \operatorname{adj}(I^d) \subseteq \operatorname{core}(I)$.

The question of equality has connections to geometry.

Kawamata's Conjecture: Let $\mathcal{L}$ be an ample line bundle on a smooth $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ such that $\mathcal{L} \otimes \omega_X^{-1}$ is also ample. Then $H^0(X, \mathcal{L}) \neq 0$.

Theorem: (Hyry, Smith) Let $R = \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{L}^n)$ and $d = \dim R$. Fix $N \gg 0$ and let $I = R_{\geq N}$. Then Kawamata's conjecture holds if and only if $\operatorname{gradedcore}(I\omega_R) = \operatorname{adj}(I^d \omega_R)$.

Partial answers to Question 1 have been found.

Theorem: (Huneke, Swanson) Let I be an integrally closed ideal in $k[[x,y]]$. Then $\operatorname{core}(I) = \operatorname{adj}(I^2)$.

Theorem: (Polini, Ulrich) If $R[It]$ has only rational singularities, then $\operatorname{core}(I) = \operatorname{adj}(I^d)$.

Theorem: (Kustin, Polini, Ulrich) Let $I \subseteq k[x,y]$ be generated by forms of the same degree. Then $\operatorname{core}(I) = \operatorname{adj}(I^2)$ if and only if $\operatorname{core}(I)$ is integrally closed.

Question 1 is addressed for monomial ideals below.

Conjecture

In addition to Assumptions, let I be a monomial ideal of R. Then $\operatorname{core}(I) = \operatorname{adj}(I^d)$ if and only if $\operatorname{core}(I)$ is integrally closed.

Consequences of the conjecture

Theorem 1: If the conjecture holds, then $R[It]$ satisfies Serre's condition R_1 if and only if $\operatorname{core}(I) = \operatorname{adj}(I^d)$.

- R_1 says a ring is locally "like" a power series ring.

Theorem 2: Let $d = 3$ and assume $I = \bar{I}$. Suppose $r(I) \leq 2$. Then I is normal if and only if $\operatorname{core}(I) = \operatorname{adj}(I^3)$.

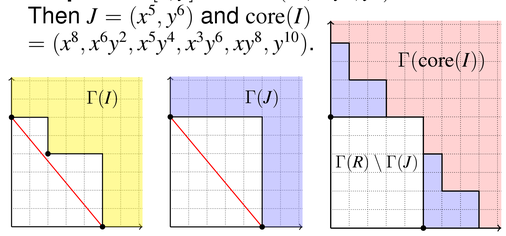
- I is **normal** if $I^n = \overline{I^n}$ for all n.
- $r(I) = \min\{r \in \mathbb{N} \mid JI^r = I^{r+1}, J \in \mathcal{M}(I)\}$ is the **reduction number** of I.

The conjecture holds in the two cases which follow.

Case 1: I has a d-generated monomial reduction J

We sandwich $\operatorname{core}(I)$ between J and $\operatorname{adj}(I^d)$ and use translational symmetry.

Example: In $k[x,y]$, let $I = (x^5, x^2y^4, y^6)$. Then $J = (x^5, y^6)$ and $\operatorname{core}(I) = (x^8, x^6y^2, x^5y^4, x^3y^6, xy^8, y^{10})$.



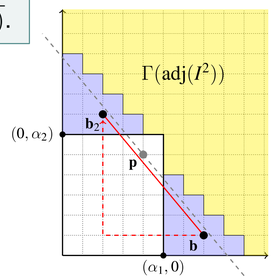
Translation Lemma: Let I be a monomial ideal of R with a reduction $J = (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})$. Let $\mathbf{b} := (b_1, \ldots, b_d) \in \mathbb{N}^d$, and assume $b_1 \geq \alpha_1$. Then for $i = 2, \ldots, d$, $\mathbf{b} \in \Gamma(\operatorname{core}(I))$ if and only if $\mathbf{b}_i \in \Gamma(I)$, where

$$\mathbf{b}_i := (b_1 - \alpha_1, b_2, \ldots, b_i + \alpha_i, b_{i+1}, \ldots, b_d).$$

Proposition 1: Let I be a monomial ideal in $k[x_1, \ldots, x_d]$ with a reduction $J = (x_1^{\alpha_1} \ldots, x_d^{\alpha_d})$. Then $\operatorname{core}(I) = \operatorname{adj}(I^d)$ if and only if $\operatorname{core}(I) = \overline{\operatorname{core}(I)}$.

Proof sketch of Proposition 1:

- We show if $\operatorname{core}(I) \neq \operatorname{adj}(I^d)$, then $\operatorname{core}(I) \neq \overline{\operatorname{core}(I)}$.
- Using Trans. Lemma, if $\mathbf{b} \in \Gamma(\operatorname{core}(I)) \setminus \Gamma(\operatorname{adj}(I^d))$, then $\mathbf{b}_i \in \Gamma(\operatorname{core}(I))$ for $i = 2, \ldots, d$.
- The point $\mathbf{p} = (\alpha_1, \ldots, \alpha_d) - (1, \ldots, 1)$ is in the convex hull of these d points. Thus, $\mathbf{p} \in \Gamma(\overline{\operatorname{core}(I)})$.
- But, $\mathbf{p} \notin \Gamma(J) \supseteq \Gamma(\operatorname{core}(I))$. Thus, $\operatorname{core}(I) \neq \overline{\operatorname{core}(I)}$.



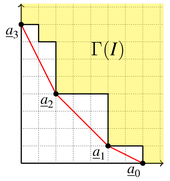
Case 2: Reduce to reduction number one in $d = 2$

We replace the J of Case 1 by an ideal C fulfilling a similar role.

Proposition 2: Let I be an $\mathfrak{m}$-primary monomial ideal in $k[x,y]$. Suppose there exists a monomial ideal L with $I \subseteq L \subsetneq \bar{I}$ and $r(L) = 1$. Then $\operatorname{core}(I) \neq \operatorname{adj}(I^2)$ and $\operatorname{core}(I) \neq \overline{\operatorname{core}(I)}$.

Given a monomial I, let $\underline{a}_0, \ldots, \underline{a}_n$ minimally generate $\mathbf{NP}(I)$, where $\underline{a}_i = (a_i, b_i)$.

Define $C := \sum_{i=1}^n (x^{a_{i-1}+a_i} y^{2b_{i-1}}) + \sum_{i=1}^n (x^{2a_i} y^{b_{i-1}+b_i})$.



Proof idea for Proposition 2:

- Consider I pictured left, and C, $\operatorname{adj}(I^2)$, and L pictured below.
- Because $r(L) = 1$, Polini and Ulrich have shown $\operatorname{core}(L) = J(J : L)$, where $J \in \mathcal{M}(L)$.
- We can obtain $\operatorname{core}(L)$ through local reflections and translations, pictured in the last two figures.
- We then use $\operatorname{core}(I) \supseteq \operatorname{core}(L)$ to prove Proposition 2.

Containment Lemma: $C \supseteq \operatorname{core}(I) \supseteq \operatorname{adj}(I^2)$.