The core and the adjoint: A condition for equality
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Assumptions and questions of research

Assumptions: Let \( R = k[x_1, \ldots, x_n] \) be a polynomial ring over a field \( k \) of characteristic zero with \( m = (x_1, \ldots, x_n) \). Let \( J \) be an \( m \)-primary ideal of \( R \) (\( m^n \not\subset J \) for some \( n \)).

Question 1: When is core of an ideal equal to the adjoint of its \( d \)-th power?

Question 2: Is there a combinatorial description for the core of a monomial ideal?

History of core and adjoint relationship

The theorem below was the original connection made between the core and the adjoint.

**Theorem:** (Lipman, generalized by Ulrich) Let \( J \) be an ideal in \( k[x_1, \ldots, x_n] \). Then \( \text{core}(J) \subseteq \text{adj}(J^d) \subseteq \text{core}(J^d) \).

The question of equality has connections to geometry.

**Kawamata's Conjecture:** Let \( L \) be an ample line bundle on a smooth \( X \subseteq \mathbb{P}^N \) such that \( L \otimes \mathcal{O}_{X,L} \) is ample. Then \( H^0(X, L^d) \neq 0 \).

**Proposition 1:** Let \( I \) be a monomial ideal in \( k[x_1, \ldots, x_n] \) with a reduction \( J = (x_1^{a_1}, \ldots, x_n^{a_n}) \).

Then \( \text{core}(I) = \text{adj}(J) \) if and only if \( \text{core}(I) = \text{adj}(J) \).

**Proof sketch of Proposition 1:**

- We show if \( \text{core}(I) = \text{adj}(J) \), then \( \text{core}(I) = \text{adj}(J) \).
- Using Trans. Lemma, if \( J = \text{core}(I) \), then \( b_i = \text{adj}(J) \), and \( I \subseteq L \).
- The point \( p = (a_1, \ldots, a_n) \) is in the convex hull of these \( d \) points. Thus, \( p \in \text{core}(I) \).
- **But,** \( p \not\in \text{adj}(I) \). Therefore, \( \text{core}(I) \neq \text{adj}(I) \).

**Case 1:** \( J \) is a \( d \)-generated monomial reduction \( J \)

We sandwich \( \text{core}(I) \) between \( J \) and \( \text{adj}(J^d) \) and use translational symmetry.

**Example:** In \( k[x, y] \), let \( I = (x^2, y^3, y^4, y^5) \). Then \( \text{core}(I) = (x, y^2, y^3, y^4, y^5) \).

**Translation Lemma:** Let \( I \) be a monomial ideal of \( R \) with a reduction \( J = (x_1^{a_1}, \ldots, x_n^{a_n}) \).

Let \( b_i = (b_{i1}, \ldots, b_{in}) \) and assume \( b_i \geq a_i \). Then for \( i = 2, \ldots, d \), \( b_i \in \text{core}(J) \) if and only if \( b_i \in \text{core}(J) \), where \( b_i = (b_{i1}, \ldots, b_{in}) \).

**Case 2:** Reduce to reduction number one in \( d = 2 \)

We replace the \( J \) of Case 1 by an ideal \( C \) fulfilling a similar role.

**Proposition 2:** Let \( I \) be an \( m \)-primary monomial ideal in \( k[x, y] \). Suppose there exists a monomial ideal \( J \) with \( I \subseteq L \subseteq J \) and \( r(J) = 1 \).

Then \( \text{core}(I) \neq \text{adj}(J) \) and \( \text{core}(I) \neq \text{adj}(J) \).

**Proof idea for Proposition 2:**

- Consider \( L \) pictured left, and \( C \) pictured below.
- Define small polynomials \( x_1^{a_1} + \cdots + x_n^{a_n} \).
- Containment Lemma: \( C \supset \text{core}(I) \).